PERGAMON

International Journal of Solids and Structures 38 (2001) 2953–2968

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Large strain viscoelastic constitutive models

J. Bonet

Department of Civil Engineering, University of Wales, Singleton Park, Swansea, Swansea SA2 8PP, UK

Received 13 June 1999; in revised form 16 March 2000

Abstract

This paper discusses a new continuum formulation for viscoelastic materials at finite strains. The model proposed is based on the multiplicative decomposition of the isochoric component of deformation gradient into elastic and viscous contribution and the generalized Maxwell rheological model. The inelastic or viscous components of the deformation gradient provide the internal variables required for the irreversible thermo-mechanical model. Nonlinear rate type of evolution equations are then proposed for the internal variables. These are based on a particular linear relaxation form of the generalized Maxwell model which leads to a viscoelastic formulation that can be seen as a particular case of a large strain viscoplastic model based on maximum plastic dissipation. In addition to the rate evolution equations, simple incremental stress update equations are proposed. These closely resemble the radial return algorithms used in von Mises plasticity. Finally a spatial form of the viscoelastic formulation is presented for isotropic materials. This formulation is based on principal directions and logarithmic stretches. Again incremental equations will be considered in order to permit subsequent computational implementations.

Keywords: Large strain; Viscoelasticity; Evolution model; Generalised Maxwell mode

1. Introduction

A number of viscoelastic formulations have been proposed in recent years in order to extend the well known linear rheological models to the large strain regime. Lubliner (1985) proposed an extension of the pioneering work of Green and Tobolsky (1946) based on the multiplicative decomposition of the deformation gradient into elastic and permanent viscous components. The resulting model is only limited by the choice of linear rate equations to describe the relaxation of the viscous strains. Unfortunately, in the finite strain range, the plastic strains are nonlinearly constrained and therefore their evolution cannot be governed by a linear rate equations. Simo (1987), Holzapfel and Reiter (1995), Holzapfel and Simo (1996), Holzapfel (1996) and Simo and Hughes (1998) have proposed alternative models in which the evolution of the viscous nonequilibrium stresses is defined directly by a linear differential equation that mimics the force relaxation process taking place in linear rheological models. These models have been extensively and successfully used in practice. However, in the common case of materials with a purely elastic volumetric response, the nonequilibrium stresses are deviatoric in nature and therefore in the material setting satisfy nonlinear constraints. Devising linear evolution equations that are compatible with such constraints is still an open question. All the above formulation have in common the separation of volumetric and isochoric components of the deformation using a multiplicative procedure discussed by Ogden (1984), Simo et al.

0020-7683/01/$ - see front matter
PII: S0020-7683(00)00215-8

(1985) and Simo and Taylor (1991). This enables the volumetric response to be purely elastic as is the case of many materials of practical interest.

The approach followed in this paper is based on the model proposed by Lubliner (1985). In common with this work the formulation is based on multiple multiplicative decompositions of the deformation gradient into viscous and elastic components. Each of these decompositions represents a spring in series with a dashpot in a linear rheological model. The permanent strain in each dashpot represents a set of internal variables required to define the thermo-dynamical state of the system. In contrast with the model of Lubliner, however, nonlinear forms of the rate equation governing the evolution of the state variables will be derived from linear relaxation laws based on a simple 1-dimensional rheological model. Significantly, this evolution equations will not only be considered in a rate form but incremental counterparts will be derived in recognition of the need for a future computational implementation. The resulting viscoelastic formulation can be seen as a particular case of the general large strain elasto-viscoplastic formulation proposed by Simo (1988) in which the elastic region has been collapsed to the origin.

The paper will in fact propose two different but related models. The first one is based in the reference configuration and is therefore suitable for any materials including those exhibiting anisotropy. Unfortunately, the incremental update of the plastic strain requires the solution of six nonlinear equations at each increment. For isotropic materials a second formulation is proposed based on principal directions of strain and logarithmic stretches. This formulation, although still based on the multiplicative decomposition and nonlinear relaxation equations, is easier to implement as it avoids the need for solving nonlinear equations to update the plastic strain. Both models will only coincide for a particular form of the strain energy functional. Alternative spatial models have been proposed by Reese and Govindjee (1998).

The formulations presented will be illustrated with the help of simple and well known hyperelastic strain energy equations, namely the compressible neo-Hookean and Hencky models (see for instance Peric et al., 1992). In addition only isothermal processes will be considered.

1.1. Rheological model

The formulation presented is based on the generalized Maxwell model, also known as Wiechert model (see for instance Findley et al., 1976), shown in Fig. 1. In this simple model, the total force includes a long term or steady state component f_∞ plus an arbitrary number of nonequilibrium forces f_α as

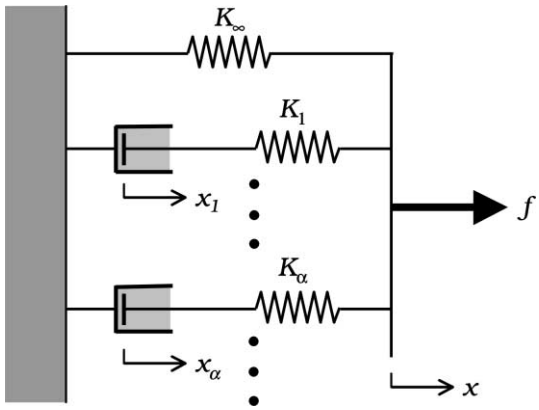

Fig. 1. Generalised Maxwell model.

$$f = f_\infty + \sum_\alpha f_\alpha. \tag{1}$$

In order to generalize the above model to the large strain 3-dimensional case, it is necessary to re-write it in terms of the total free energy of the system. As in the case of the force, this is achieved by simply adding the steady state and nonequilibrium terms to give,

$$\Psi(x) = \Psi_\infty(x) + \sum_\alpha \Psi_\alpha(x, x_\alpha), \tag{2}$$

where for the simple linear model described in Fig. 1, the strain energy terms are expressed in terms of the corresponding strain in the springs and the spring constants as

$$\Psi_\infty = \tfrac{1}{2} K_\infty x^2, \tag{3a}$$

$$\Psi_\alpha = \tfrac{1}{2} K_\alpha (x - x_\alpha)^2. \tag{3b}$$

Note that in this equation, the displacements in the viscous elements are internal variables which must be known in order to fully describe the state of the system. Differentiating Eq. (2) with respect to the total strain variable x gives the total force in the system as

$$f = K_\infty x + \sum_\alpha K_\alpha (x - x_\alpha). \tag{4}$$

The above model is not complete without equations describing the evolution of the internal variables x_α. For each of the viscous elements depicted in Fig. 1, this is defined by a relationship between the force in the corresponding spring and the rate of change of the viscous deformation as

$$c_\alpha \dot{x}_\alpha = f_\alpha, \tag{5a}$$

$$f_\alpha = K_\alpha (x - x_\alpha), \tag{5b}$$

where c_α represents the linear viscosity of the dashpot and can be related to the stiffness of the corresponding spring in terms of a *retardation time* parameter τ_α as

$$c_\alpha = \tau_\alpha K_\alpha. \tag{6}$$

This definition enables Eqs. (5a) and (5b) to be re-written in the form of an evolution equation for the internal variable x_α as

$$\dot{x}_\alpha = \frac{1}{\tau_\alpha}(x - x_\alpha). \tag{7}$$

A nonlinear extension of Eq. (7) will be used to define the evolution of the internal variables in the spatial model proposed in Section 3. However, the extension of Eq. (7) for the nonlinear Lagrangian formulation is more difficult. This is due to the fact that the internal variables to be used will be nonlinearly constrained and therefore unable to satisfy a simple linear evolution law as given by Eq. (7). An alternative form of Eq. (7), more suitable for extension to the Lagrangian nonlinear regime is obtained by expressing Eq. (5a) in terms of the relaxation of nonequilibrium forces f_α at constant total strain x to give

$$\left.\frac{\mathrm{d}f_\alpha}{\mathrm{d}t}\right|_{x=\text{constant}} = -\frac{1}{\tau_\alpha} f_\alpha. \tag{8}$$

It is a trivial exercise to show that in the linear model described above, Eq. (8) combined with Eqs. (5b) and (6) leads immediately to Eq. (7). Similarly, in the nonlinear context an obvious generalization of Eq. (8) will lead to correct evolution equations for the nonlinear internal variables used.

2. Material (Lagrangian) formulation

Many viscoelastic materials of practical interest, such as human body soft tissues, exhibit anisotropic behaviour, Holzapfel et al. (1996), Bonet and Burton (1998). In such cases the constitutive model has to be defined in the material or reference configuration in terms of second Piola–Kirchhoff stresses and right Cauchy–Green or similar strain measures. The resulting equations can be subsequently pushed forward to the current or spatial configuration to define Cauchy stresses. This section will describe a nonlinear viscoelastic model in the reference configuration which is therefore suitable for anisotropic materials.

2.1. Multiplicative decomposition

Consider the motion $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X}, t)$ of a general body in three dimensions so that a particle at initial position $\boldsymbol{X}$ occupies position $\boldsymbol{x}$ at time t. Let also $\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{\phi}$ denote the deformation gradient of such motion at a given instant in time. In common with current large strain elasto-plastic models, the kinematic foundation of the model proposed is the multiplicative decomposition of the deformation gradient $\boldsymbol{F}$ into volumetric, isochoric elastic and viscous components. In order to define this process, consider first the decomposition of the deformation gradient $\boldsymbol{F}$ of a general 3-dimensional motion in terms of volumetric and volume preserving (isochoric) components as (Simo et al., 1985),

$$\boldsymbol{F} = J^{1/3}\widehat{\boldsymbol{F}}; \quad J = \det \boldsymbol{F}. \tag{9}$$

By construction the tensor $\widehat{\boldsymbol{F}}$ is isochoric, whereas the change in volume during the motion is given by the Jacobian J. For most of metals and polymer based materials, the volumetric deformation is purely elastic and the viscous effects are restricted to the isochoric component of the deformation. The above decomposition easily permits volume and volume preserving parts to be treated separately.

In order to extend the general Maxwell model described in Fig. 1 to the large strain 3-dimensional regime, consider the multiplicative decomposition of the isochoric deformation gradient into *multiple* elastic and permanent or viscous components as (Lubliner, 1985)

$$\widehat{\boldsymbol{F}} = \widehat{\boldsymbol{F}}_{\mathrm{e}_\alpha}\boldsymbol{F}_{\mathrm{v}_\alpha}. \tag{10}$$

This equation is illustrated in Fig. 2. As in the generalized Maxwell model, the number of viscous components α is arbitrary. Each viscous element has an associated tensor of internal variables $\boldsymbol{F}_{\mathrm{v}_\alpha}$ which defines the current state of the system. Thermo-dynamical equilibrium is achieved when $\boldsymbol{F}_{\mathrm{v}_\alpha} = \widehat{\boldsymbol{F}}$. Note that the assumed isochoric nature of the viscous deformation implies that $\det \boldsymbol{F}_{\mathrm{v}_\alpha} = 1$.

In order to ensure that the model is independent of rigid body rotations in the spatial configuration, the total, isochoric, elastic, and permanent Cauchy–Green tensor strain tensors corresponding to the above deformation gradient components are defined as

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}, \tag{11a}$$

$$\widehat{\boldsymbol{C}} = \widehat{\boldsymbol{F}}^{\mathrm{T}}\widehat{\boldsymbol{F}} = I_3^{-1/3}\boldsymbol{C}; \quad I_3 = \det \boldsymbol{C} = J^2, \tag{11b}$$

$$\widehat{\boldsymbol{C}}_{\mathrm{e}_\alpha} = \widehat{\boldsymbol{F}}_{\mathrm{e}_\alpha}^{\mathrm{T}}\widehat{\boldsymbol{F}}_{\mathrm{e}_\alpha}, \tag{11c}$$

$$\boldsymbol{C}_{\mathrm{v}_\alpha} = \boldsymbol{F}_{\mathrm{v}_\alpha}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{v}_\alpha}. \tag{11d}$$

The above tensors, rather than the deformation gradients will be used as main variables in the Lagrangian formulation in order to ensure objectivity. Observe, however, that unlike $\boldsymbol{C}$, $\widehat{\boldsymbol{C}}$ and $\boldsymbol{C}_{\mathrm{v}_\alpha}$, the tensor $\widehat{\boldsymbol{C}}_{\mathrm{e}_\alpha}$ is not based at the reference configuration but at the unloaded state. For this reason, J, $\widehat{\boldsymbol{C}}$ and $\boldsymbol{C}_{\mathrm{v}_\alpha}$ are the set of variables used to define the thermo-dynamical state. Note also that the condition $\det \boldsymbol{F}_{\mathrm{v}_\alpha} = 1$ implies $\det \boldsymbol{C}_{\mathrm{v}_\alpha} = 1$.

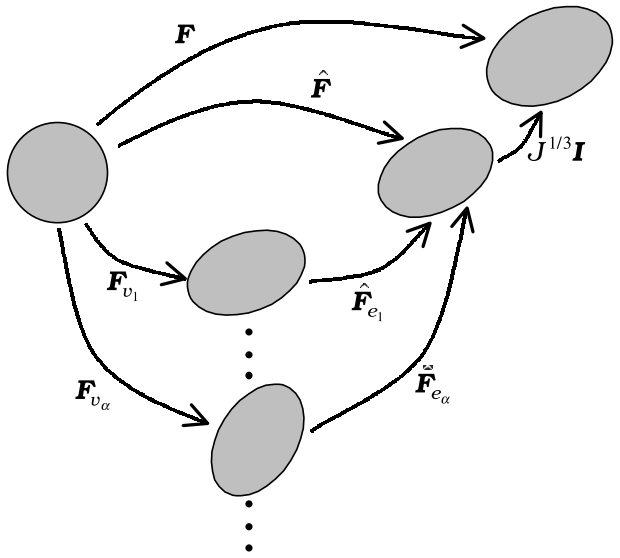

Fig. 2. Multiplicative decomposition.

2.2. Strain energy

The starting point of the nonlinear model is an expression for the elastic energy of the system. In common with Lubliner (1985), Holzapfel and Reiter (1995) and Holzapfel and Simo (1996) and by analogy with the Maxwell equation (2), this is assumed to be given as the sum of volumetric, long term and viscous components as

$$\Psi(\boldsymbol{C}, \boldsymbol{C}_{v_\alpha}) = U(J) + \Psi_\infty(\widehat{\boldsymbol{C}}) + \sum_\alpha \Psi_\alpha(\widehat{\boldsymbol{C}}, \boldsymbol{C}_{v_\alpha}). \tag{12}$$

Note that the nonequilibrium viscous terms are functions of both $\widehat{\boldsymbol{C}}$ and $\boldsymbol{C}_{v_\alpha}$ so that Ψ_α can vanish as $\boldsymbol{C}_{v_\alpha}$ approaches $\widehat{\boldsymbol{C}}$. A typical example of the strain energy function is the simple compressible neo-Hookean material defined by volumetric and isochoric components given by (see for instance Bonet and Wood, 1997),

$$U(J) = \tfrac{1}{2}K(J-1)^2, \tag{13}$$

$$\Psi_\infty = \tfrac{1}{2}\mu(\operatorname{tr}\widehat{\boldsymbol{C}} - 3) = \tfrac{1}{2}\mu(I_3^{-1/3}\boldsymbol{C} : \boldsymbol{I} - 3). \tag{14}$$

In addition, again simply as an illustration of Eq. (12), the strain energy function associated to each of the viscous components can be assumed to be proportional to the long term expression and therefore given as (Govindjee and Simo, 1992),

$$\Psi_\alpha(\widehat{\boldsymbol{C}}, \boldsymbol{C}_{v_\alpha}) = \beta_\alpha \Psi_\infty(\widehat{\boldsymbol{C}}_{e_\alpha}), \tag{15}$$

where β_α are positive nondimensional proportionality factors. For the case of the neo-Hookean model, the above assumption combined with the fact that the invariants of $\widehat{\boldsymbol{C}}_{e_\alpha}$ coincide with the invariants of $\widehat{\boldsymbol{C}}\boldsymbol{C}_{v_\alpha}^{-1}$ leads to

$$\Psi_\alpha(\widehat{\boldsymbol{C}}, \boldsymbol{C}_{v_\alpha}) = \tfrac{1}{2}\beta_\alpha\mu(\widehat{\boldsymbol{C}} : \boldsymbol{C}_{v_\alpha}^{-1} - 3). \tag{16}$$

Note, however, that Eq. (15) is only strictly correct for materials where the permanent strain energy function Ψ_∞ is isotropic, that is a function of $\widehat{\boldsymbol{C}}$ via its invariants. This is of course the case for the illustrative neo-Hookean material chosen above, but it will not apply to more general anisotropic constitutive models.

2.3. Second Piola–Kirchhoff stress tensor

The second Piola–Kirchhoff stress tensor $\boldsymbol{S}$ can now be obtained by differentiating the strain energy function with respect to the right Cauchy–Green tensor $\boldsymbol{C}$. Given the de-coupled nature of the strain energy equation (12), this derivative leads to volumetric and 'true' deviatoric components of $\boldsymbol{S}$ as

$$\boldsymbol{S} = 2\frac{\partial \Psi}{\partial \boldsymbol{C}} = \boldsymbol{S}_{\text{vol}} + \boldsymbol{S}', \tag{17a}$$

$$\boldsymbol{S}_{\text{vol}} = \frac{\partial U}{\partial \boldsymbol{C}}, \tag{17b}$$

$$\boldsymbol{S}' : \boldsymbol{C} = 0. \tag{17c}$$

The volumetric component of $\boldsymbol{S}$ can be expressed in terms of the isotropic pressure p by noting that $J^2 = I_3 = \det \boldsymbol{C}$ and making use of the following relationship (see for instance Bonet and Wood, 1997):

$$\frac{\partial I_3}{\partial \boldsymbol{C}} = I_3 \boldsymbol{C}^{-1} \tag{18}$$

to give after simple algebra

$$\boldsymbol{S}_{\text{vol}} = pJ\boldsymbol{C}^{-1}, \quad p = \frac{\mathrm{d}U}{\mathrm{d}J}. \tag{19}$$

The 'true' deviatoric component of $\boldsymbol{S}$ satisfies condition (17c) rather than having a trace equal to zero. In effect, the product $\boldsymbol{S} : \boldsymbol{C}$ can be interpreted as the trace with respect to the metric tensor $\boldsymbol{C}$. In order to evaluate $\boldsymbol{S}'$, the isochoric component of the strain energy equation (12) needs to be differentiated with respect to $\boldsymbol{C}$ to give

$$\boldsymbol{S}' = \boldsymbol{S}'_\infty + \sum_\alpha \boldsymbol{S}'_\alpha, \tag{20}$$

where the long term and nonequilibrium components of the deviatoric second Piola–Kirchhoff tensor are given as

$$\boldsymbol{S}'_\infty = 2\frac{\partial \Psi_\infty(\widehat{\boldsymbol{C}})}{\partial \boldsymbol{C}}, \tag{21a}$$

$$\boldsymbol{S}'_\alpha = 2\frac{\partial \Psi_\alpha(\widehat{\boldsymbol{C}}, \boldsymbol{C}_{\text{v}\alpha})}{\partial \boldsymbol{C}}. \tag{21b}$$

As an illustration of the general set of Eqs. (19)–(21a) and (21b), consider again the compressible neo-Hookean case with proportional nonequilibrium components. For this simple example, Eqs. (19) and (13) yield the pressure as

$$p = K(J-1). \tag{22}$$

Differentiating Eq. (14) with the help of expression (18) gives the equilibrium deviatoric second Piola–Kirchhoff stress component as

$$\boldsymbol{S}'_{\infty} = \mu I_3^{-1/3}\left(\boldsymbol{I} - \tfrac{1}{3} I_1 \boldsymbol{C}^{-1}\right), \tag{23a}$$

$$I_1 = \operatorname{tr} \boldsymbol{C}. \tag{23b}$$

Finally, an identical derivation using Eqs. (21b) and (16) gives the nonequilibrium components for this simply neo-Hookean model as

$$\boldsymbol{S}'_{\alpha} = \beta_{\alpha} \mu I_3^{-1/3}\left[\boldsymbol{C}_{\mathrm{v}_{\alpha}}^{-1} - \tfrac{1}{3}(\boldsymbol{C} : \boldsymbol{C}_{\mathrm{v}_{\alpha}}^{-1})\boldsymbol{C}^{-1}\right]. \tag{24}$$

Note that as $\boldsymbol{C}_{\mathrm{v}_{\alpha}}$ approaches $\widehat{\boldsymbol{C}} = I_3^{-1/3}\boldsymbol{C}$, the above tensor vanishes, in the same way as in the rheological model the spring force f_{α} tends to zero as the strain x_{α} catches up with the total strain x.

2.4. Evolution equation – rate form

In order to complete the description of the viscoelastic model proposed, it is now necessary to define equations for the evolution of the set of internal variables embodied in the viscous strain tensors $\boldsymbol{C}_{\mathrm{v}_{\alpha}}$. In Lubliner (1985), simple linear equations similar to Eq. (7) for the rheological are proposed. However, these linear relationships are incompatible with the nonlinear condition $\det \boldsymbol{C}_{\mathrm{v}_{\alpha}} = 1$. In contrast, in Simo (1987), Holzapfel and Simo (1996) and Holzapfel (1996) simple linear evolution laws are assumed for the non-equilibrium forces $\boldsymbol{S}'_{\alpha}$ directly. Again this presents the problem of ensuring that the condition $\boldsymbol{S}'_{\alpha} : \boldsymbol{C} = 0$ is not violated. The approach followed here, albeit similar, is based on the relaxation form of the viscous forces contained in Eq. (8). It is in fact a simple step to extend this equation to the nonlinear regime by simply replacing the forces by the second Piola–Kirchhoff tensor and the strain by the right Cauchy–Green tensor to give,

$$\left.\frac{\mathrm{d}\boldsymbol{S}'_{\alpha}}{\mathrm{d}t}\right|_{\boldsymbol{C}=\text{constant}} = -\frac{1}{\tau_{\alpha}}\boldsymbol{S}'_{\alpha}, \tag{25}$$

where τ_{α} is again a retardation or relaxation time parameter that will determine the rate of dissipation of viscous stresses. Given that $\boldsymbol{S}'_{\alpha}$ is a function of both $\boldsymbol{C}$ and $\boldsymbol{C}_{\mathrm{v}_{\alpha}}$ and using the chain rule, the left-hand side derivative in this equation can be expressed as

$$\frac{\partial \boldsymbol{S}'_{\alpha}}{\partial \boldsymbol{C}_{\mathrm{v}_{\alpha}}} : \dot{\boldsymbol{C}}_{\mathrm{v}_{\alpha}} = -\frac{1}{\tau_{\alpha}}\boldsymbol{S}'_{\alpha}, \tag{26}$$

thereby leading to the following nonlinear evolution equation for the state variables $\boldsymbol{C}_{\mathrm{v}_{\alpha}}$,

$$\dot{\boldsymbol{C}}_{\mathrm{v}_{\alpha}} = -\frac{1}{\tau_{\alpha}}\mathbf{M}_{\alpha}^{-1} : \boldsymbol{S}'_{\alpha}, \tag{27a}$$

$$\mathbf{M}_{\alpha} = 2\frac{\partial^2 \Psi_{\alpha}(\widehat{\boldsymbol{C}}, \boldsymbol{C}_{\mathrm{v}_{\alpha}})}{\partial \boldsymbol{C} \partial \boldsymbol{C}_{\mathrm{v}_{\alpha}}}. \tag{27b}$$

A number of interesting observation must be made in relation to this equation. Firstly, note that it closely resembles the evolution equation given by Simo (1988) for the plastic strain in the context of a large strain elastoplastic formulation based on the maximum plastic dissipation. In fact, it is easy to show that Eqs. (27a) and (27b) corresponds to a viscoplastic case in which the yield surface $\phi(\boldsymbol{C}, \boldsymbol{C}_p)$ is defined by the nonequilibrium strain energy function and the elastic region has been collapsed to the origin. This similarity can be used to generalize Eq. (25) by simply replacing the right-hand side with an appropriate flow rule. In this way a fully nonlinear evolution model could be constructed. The crucial difference between Eq. (25) and the similar models proposed by Simo (1987), Holzapfel and Simo (1996) and Holzapfel (1996), is the

fact that the time derivative is now taken at *constant* total deformation $\boldsymbol{C}$. This makes the evolution equation perfectly compatible with the constraint $\boldsymbol{S}'_\alpha : \boldsymbol{C} = 0$. Finally, it must be emphasized that although the evolution equations (27a) and (27b) for the state variables $\boldsymbol{C}_{v_\alpha}$ is nonlinear, the underlying relaxation law embodied by Eq. (25) is clearly linear. A detailed account of fully nonlinear evolution models is given by Lion (1997).

2.5. Incremental evolution equation

In the context of a computational model, Eqs. (27a) and (27b) would have to be integrated in time in order to update the state variables from step n to step $n+1$. However, the use of this equation is highly cumbersome and can be easily avoided by directly re-interpreting the left-hand side of Eq. (25) in an incremental form as

$$\left.\frac{\mathrm{d}\boldsymbol{S}'_\alpha}{\mathrm{d}t}\right|_{\boldsymbol{C}=\text{constant}} = \frac{1}{\Delta t}\left[\boldsymbol{S}'_\alpha\left(\boldsymbol{C}_{n+1}, \boldsymbol{C}^{n+1}_{v_\alpha}\right) - \boldsymbol{S}'_\alpha\left(\boldsymbol{C}_{n+1}, \boldsymbol{C}^{n}_{v_\alpha}\right)\right], \tag{28}$$

where Δt denotes the timestep $t_{n+1} - t_n$. Note that the term $\boldsymbol{S}'_\alpha(\boldsymbol{C}_{n+1}, \boldsymbol{C}^n_{v_\alpha})$ represents the new state of stresses that would be obtained in the absence of further viscous deformation. It can also be interpreted as the stresses that would result if the deformation from step n to step $n+1$ took place instantaneously. In the context of elastoplasticity, this term is often known as the *trial* state of stresses which may or may not be compatible with the yield surface inequality. In the present viscoelastic context a relaxation of this stress is inevitable and the final stresses can be derived by substituting Eq. (28) back into Eq. (25) to give

$$\boldsymbol{S}'_\alpha\left(\boldsymbol{C}_{n+1}, \boldsymbol{C}^{n+1}_{v_\alpha}\right) - \boldsymbol{S}'\left(\boldsymbol{C}_{n+1}, \boldsymbol{C}^{n}_{v_\alpha}\right) = -\frac{\Delta t}{\tau_\alpha}\,\boldsymbol{S}'_\alpha\left(\boldsymbol{C}_{n+1}, \boldsymbol{C}^{n+1}_{v_\alpha}\right). \tag{29}$$

Note that the right-hand side of the equation has been taken at step $n+1$. This coincides with a backward Euler (or fully implicit) integration of Eq. (25) which is only first order accurate in time. Although more accurate integration rules are possible, this form has the advantage of leading to a particularly simple form of the stress update obtained by re-arranging terms in Eq. (29) to give

$$\boldsymbol{S}'^{(n+1)}_\alpha = \frac{\tau_\alpha}{\tau_\alpha + \Delta t}\,\widetilde{\boldsymbol{S}}'^{(n+1)}_\alpha, \tag{30a}$$

$$\widetilde{\boldsymbol{S}}'^{(n+1)}_\alpha = \boldsymbol{S}'_\alpha\left(\boldsymbol{C}_{n+1}, \boldsymbol{C}^{n}_{v_\alpha}\right). \tag{30b}$$

In this expression, the final relaxed stresses are simply proportional to the instantaneous stresses $\widetilde{\boldsymbol{S}}'_\alpha$. Again borrowing the terminology of elastoplasticity, Eq. (30a) can be interpreted as a generalized radial return rule. Obviously, the above equation is much easier to implement in a computational context than the integration of Eqs. (27a) and (27b).

Equation (30a) is deceptively simple. In fact, once the nonequilibrium stresses are evaluated, it is necessary to obtain the new state variables at step $n+1$. This is not a trivial matter as it implies inverting expression (21b), or in practical terms solving for $\boldsymbol{C}_{v_\alpha}$ from the set of six nonlinear equations $\boldsymbol{S}'_\alpha(\boldsymbol{C}_{n+1}, \boldsymbol{C}^{n+1}_{v_\alpha}) = \boldsymbol{S}'^{(n+1)}_\alpha$ with the additional condition $\det \boldsymbol{C}_{v_\alpha} = 1$. (Note that this additional condition is essential as the equations are not fully independent given that $\boldsymbol{C} : \boldsymbol{S}' = 0$.) Hence, despite using the fundamentally linear equation (25) to construct an incremental evolution algorithm, the underlying nonlinearity of the evolution equations (27a) and (27b) for the state state variables $\boldsymbol{C}_{v_\alpha}$, is not entirely avoided and it re-emerges in the form of a nonlinear set of equations for the values of the new state variables. For instance, in the particular case of the neo-Hookean model discussed above, Eq. (24) gives the new internal variables in terms of the stresses as

$$(\boldsymbol{C}_{v_\alpha}^{n+1})^{-1} = \frac{1}{\beta_\alpha \mu I_3^{-1/3}} \boldsymbol{S}_\alpha'^{(n+1)} + \lambda \boldsymbol{C}_{n+1}^{-1}, \tag{31}$$

where $\lambda = \boldsymbol{C} : \boldsymbol{C}_{v_\alpha}^{-1}$ is an additional unknown to be solved for using the condition $\det \boldsymbol{C}_{v_\alpha} = 1$. This gives a nonlinear equation for λ as

$$\det\left[\boldsymbol{A} + \lambda \boldsymbol{C}_{n+1}^{-1}\right] = 1, \tag{32a}$$

$$\boldsymbol{A} = \frac{1}{\beta_\alpha \mu I_3^{-1/3}} \boldsymbol{S}_\alpha'^{(n+1)} \tag{32b}$$

which can be solved using a simple Newton–Raphson type of iteration.

2.6. Tangent modulus

The use of the above viscoelastic model in the context of a finite element formulation will require the derivation of a tangent material operator in order to permit the use of a global Newton–Raphson iteration scheme (e.g. Bonet and Wood, 1997). This operator measure the changes in stress resulting from changes in total strain and in the present Lagrangian or material setting is defined as

$$\mathsf{C} = 2\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} = 2\frac{\partial (pJ\boldsymbol{C}^{-1})}{\partial \boldsymbol{C}} + 2\frac{\partial \boldsymbol{S}'}{\partial \boldsymbol{C}}. \tag{33}$$

The volumetric component is independent of the current viscoelastic formulation and can be easily derived with the help of Eq. (18) to give (Bonet and Wood, 1997),

$$\mathsf{C}_{\text{vol}} = 2\frac{\partial (pJ\boldsymbol{C}^{-1})}{\partial \boldsymbol{C}} = (KJ^2 + pJ)(\boldsymbol{C}^{-1} \otimes \boldsymbol{C}^{-1}) - 2pJ(\boldsymbol{C}^{-1} \odot \boldsymbol{C}^{-1}), \quad K = U''(J), \tag{34}$$

where K denotes the tangent bulk modulus of the material and the tensor notation $\boldsymbol{C}^{-1} \odot \boldsymbol{C}^{-1}$ introduced by Holzapfel (1996) has been used to represent a fourth order tensor defined as

$$\boldsymbol{C}^{-1} \odot \boldsymbol{C}^{-1} = -\frac{\partial \boldsymbol{C}^{-1}}{\partial \boldsymbol{C}}, \tag{35a}$$

$$2(\boldsymbol{C}^{-1} \odot \boldsymbol{C}^{-1})_{IJKL} = (\boldsymbol{C}^{-1})_{IK}(\boldsymbol{C}^{-1})_{JL} + (\boldsymbol{C}^{-1})_{IL}(\boldsymbol{C}^{-1})_{JK}. \tag{35b}$$

The deviatoric components of the tangent operator are expressed in terms of long term and nonequilibrium contributions as

$$\widehat{\mathsf{C}} = 2\frac{\partial \boldsymbol{S}'}{\partial \boldsymbol{C}} = \widehat{\mathsf{C}}_\infty + \sum_\alpha \widehat{\mathsf{C}}_\alpha, \tag{36a}$$

$$\widehat{\mathsf{C}}_\infty = 2\frac{\partial \boldsymbol{S}'_\infty}{\partial \boldsymbol{C}}, \tag{36b}$$

$$\widehat{\mathsf{C}}_\alpha = 2\frac{\partial \boldsymbol{S}'_\alpha}{\partial \boldsymbol{C}}. \tag{36c}$$

For instance for the neo-Hookean case, standard algebra leads to a steady state component given as

$$\widehat{\mathsf{C}}_\infty = 2\mu I_3^{-1/3}\left(\tfrac{1}{3}I_1 \boldsymbol{C}^{-1} \odot \boldsymbol{C}^{-1} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{C}^{-1} - \tfrac{1}{3}\boldsymbol{C}^{-1} \otimes \boldsymbol{I} + \tfrac{1}{9}I_1 \boldsymbol{C}^{-1} \otimes \boldsymbol{C}^{-1}\right). \tag{37}$$

Finally, the evaluation of the nonequilibrium contribution could be far more complicated as the changes in the internal variables $\boldsymbol{C}_{v_\alpha}$ arising from a change in $\boldsymbol{C}$ would also need to be taken into account.

Fortunately, the simplicity of Eq. (30a) makes this process unnecessary as clearly the differentiation of the final relaxed stresses is given in terms of the derivatives of the instantaneous stress tensor as,

$$\widehat{\mathbf{C}}_\alpha = \frac{\tau_\alpha}{\tau_\alpha + \Delta t}\,\widetilde{\mathbf{C}}_\alpha, \tag{38a}$$

$$\widetilde{\mathbf{C}}_\alpha = \frac{\partial \widetilde{\boldsymbol{S}}'^{(n+1)}_\alpha}{\partial \boldsymbol{C}_{n+1}}. \tag{38b}$$

Again in the case of a proportional neo-Hookean equation, the differentiation of Eq. (24), taking the total right Cauchy–Green tensor $\boldsymbol{C}$ at $n+1$ whilst $\boldsymbol{C}_{\mathrm{v}_\alpha}$ is fixed at n gives

$$\begin{aligned}\widetilde{\mathbf{C}}^{n+1}_\alpha = \frac{2\mu\beta_\alpha\tau_\alpha I_3^{-1/3}}{\tau_\alpha+\Delta t}\Big[&\tfrac{1}{3}\big(\boldsymbol{C}_{n+1} : \boldsymbol{C}^{-1\,(n)}_{\mathrm{v}_\alpha}\big)\boldsymbol{C}^{-1}_{n+1}\odot\boldsymbol{C}^{-1}_{n+1} - \tfrac{1}{3}\boldsymbol{C}^{-1\,(n)}_{\mathrm{v}_\alpha}\otimes\boldsymbol{C}^{-1}_{n+1} - \tfrac{1}{3}\boldsymbol{C}^{-1}_{n+1}\otimes\boldsymbol{C}^{-1\,(n)}_{\mathrm{v}_\alpha}\\ &+\tfrac{1}{9}\big(\boldsymbol{C}_{n+1} : \boldsymbol{C}^{-1\,(n)}_{\mathrm{v}_\alpha}\big)\boldsymbol{C}^{-1}_{n+1}\otimes\boldsymbol{C}^{-1}_{n+1}\Big].\end{aligned} \tag{39}$$

3. Spatial formulation

The above Lagrangian formulation is suitable for all types of materials, including those exhibiting anisotropic behaviour. For isotropic materials, however, it is possible and often preferred to employ a spatial formulation to describe the state of stress and strain Simo (1992), Peric et al. (1992), Miehe (1994). Typically, the Cauchy stress tensor $\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{\mathrm{T}}$ and the left Cauchy–Green (or Finger) strain $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ are used for this purpose. In the context of the multiplicative decomposition, the isochoric and elastic components of the left Cauchy–Green tensors are given as

$$\hat{\boldsymbol{b}} = \widehat{\boldsymbol{F}}\,\widehat{\boldsymbol{F}}^{\mathrm{T}}, \tag{40a}$$

$$\hat{\boldsymbol{b}}_{\mathrm{e}_\alpha} = \widehat{\boldsymbol{F}}_{\mathrm{e}_\alpha}\widehat{\boldsymbol{F}}^{\mathrm{T}}_{\mathrm{e}_\alpha}. \tag{40b}$$

Note that both these tensors are based on the current spatial configuration. The strain energy can now be expressed as a function of J, $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{b}}_{\mathrm{e}_\alpha}$ to give

$$\Psi(\boldsymbol{b}, \hat{\boldsymbol{b}}_{\mathrm{e}_\alpha}) = U(J) + \Psi_\infty(\hat{\boldsymbol{b}}) + \sum_\alpha \Psi_\alpha(\hat{\boldsymbol{b}}_{\mathrm{e}_\alpha}). \tag{41}$$

For many isotropic materials, the dependency of the strain energy on $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{b}}_{\mathrm{e}_\alpha}$ is often expressed in terms of the principal stretches of the deformation λ_i ($i = 1,2,3$) defined as the square root of the eigenvalues of $\boldsymbol{b}$ (see for instance Ogden, 1984). Similarly, the total isochoric stretches $\hat{\lambda}_i$ and their elastic counterparts $\hat{\lambda}^{\mathrm{e}_\alpha}_i$ can be evaluated from the expressions

$$\hat{\boldsymbol{b}}\boldsymbol{n}_i = \hat{\lambda}^2_i\boldsymbol{n}_i, \quad \hat{\lambda}_i = J^{-1/3}\lambda_i, \tag{42a}$$

$$\hat{\boldsymbol{b}}_{\mathrm{e}_\alpha}\boldsymbol{n}_{i\alpha} = (\hat{\lambda}^{\mathrm{e}_\alpha}_i)^2\boldsymbol{n}_{i\alpha}, \quad \hat{\lambda}^{\mathrm{e}_\alpha}_i = J^{-1/3}\lambda^{\mathrm{e}_\alpha}_i, \tag{42b}$$

where $\boldsymbol{n}_i$ are the principal directions of the deformation of $\hat{\boldsymbol{b}}$ and $\boldsymbol{n}_{i\alpha}$ the principal directions of $\hat{\boldsymbol{b}}_{\mathrm{e}_\alpha}$. Note, that these principal directions will, in general, be different. Note also that $J = \lambda_1\lambda_2\lambda_3$ and, by construction, the isochoric stretches satisfy $\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3 = 1$. It is now possible to re-write the strain energy equation (41) as a function of the total and elastic stretches to give

$$\Psi(J, \hat{\lambda}_i, \hat{\lambda}^{\mathrm{e}_\alpha}_i) = U(J) + \Psi_\infty(\hat{\lambda}_i) + \sum_\alpha \Psi_\alpha(\hat{\lambda}^{\mathrm{e}_\alpha}_i). \tag{43}$$

3.1. Cauchy stress

In the same way as the second Piola–Kirchhoff is split into volumetric, long term isochoric and non-equilibrium components, the Cauchy stress tensor will now be similarly decomposed as

$$\boldsymbol{\sigma} = p\boldsymbol{I} + \boldsymbol{\sigma}'_{\infty} + \sum_{\alpha} \boldsymbol{\sigma}'_{\alpha}, \tag{44}$$

where as in Eq. (19) the pressure is given as the derivative of the volumetric energy function U and the " $'$ " symbol has now the usual meaning of standard deviatoric component of the corresponding tensor. The isotropic nature of the materials under consideration implies that the above stresses share the same principal directions as their corresponding strain tensors and therefore can be expressed as

$$\boldsymbol{\sigma}'_{\infty} = \sum_{i=1}^{3} \sigma'_{i\infty} \boldsymbol{n}_i \otimes \boldsymbol{n}_i, \tag{45a}$$

$$\boldsymbol{\sigma}'_{\alpha} = \sum_{i=1}^{3} \sigma'_{i\alpha} \boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{i\alpha}, \tag{45b}$$

where the principal stress components can be obtained by differentiation of the strain energy terms with respect to logarithmic stretches to give (see Chapter 5 of Bonet and Wood, 1997 for a more detailed explanation)

$$\sigma'_{i\infty} = \frac{1}{J} \frac{\partial \Psi_{\infty}}{\partial \ln \lambda_i}, \tag{46a}$$

$$\sigma'_{i\alpha} = \frac{1}{J} \frac{\partial \Psi_{\alpha}}{\partial \ln \lambda_i^{e_\alpha}}. \tag{46b}$$

As a simple illustration of the above equations, consider the material commonly used to describe metals in the large strain elastoplastic range (Simo, 1992; Peric et al., 1992) in which the strain energy is defined as

$$\Psi_{\infty} = \mu \sum_{i=1}^{3} (\ln \hat{\lambda}_i)^2 = \mu \sum_{i=1}^{3} (\ln \lambda_i)^2 - \frac{1}{3}\mu \left(\sum_{i=1}^{3} \ln \lambda_i \right)^2, \tag{47a}$$

$$\Psi_{\alpha} = \beta_{\alpha}\mu \sum_{i=1}^{3} (\ln \hat{\lambda}_i^{e_\alpha})^2 = \beta_{\alpha}\mu \sum_{i=1}^{3} (\ln \lambda_i^{e_\alpha})^2 - \frac{1}{3}\beta_{\alpha}\mu \left(\sum_{i=1}^{3} \ln \lambda_i^{e_\alpha} \right)^2, \tag{47b}$$

where again the strain energy Ψ_{α} associated to each viscous component α has been assumed proportional to the long term function Ψ_{∞}. Differentiating with respect to the logarithmic stretches gives the principal deviatoric stress components as

$$J\sigma'_{i\infty} = 2\mu \ln \hat{\lambda}_i, \tag{48a}$$

$$J\sigma'_{i\alpha} = 2\mu\beta_{\alpha} \ln \hat{\lambda}_i^{e_\alpha}. \tag{48b}$$

Note that these equations are identical to those employed in linear elasticity except for the use of logarithmic strains instead of small strains.

3.2. Incremental kinematics

In order to employ the equations described in the previous section at a given stage of the deformation process, it is first necessary to be able to evaluate the elastic deformation gradient corresponding to each

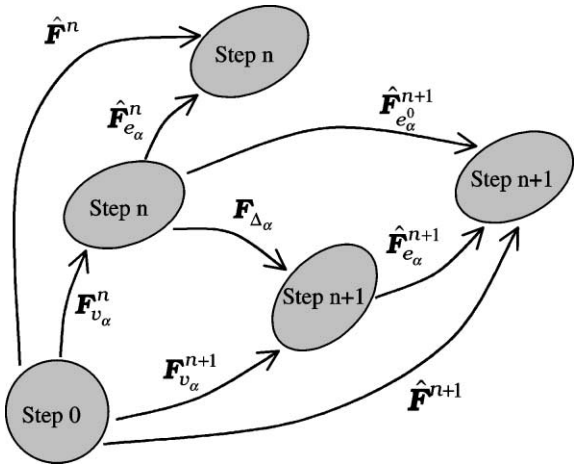

Fig. 3. Multiplicative decomposition at steps n and $n+1$.

viscous element α. This cannot be achieved, however, without the knowledge of the permanent strain accumulated in the viscous elements. In order to derive some evolution equations for the accumulated viscous strain in the spatial configuration, it is first necessary to consider the incremental motion from step n to step $n+1$ and the multiplicative decompositions of the respective deformation gradient tensors. This process is shown in Fig. 3 for a single Maxwell element α. Note that no assumptions are made about the size of this incremental motion, although in practice, small but finite increments will be required for an accurate simulation. If the deformation from stage n to $n+1$ took place instantly, without any time for viscous relaxation, the resulting elastic state would be described by the deformation gradient $\widehat{\boldsymbol{F}}^{n+1}_{\mathrm{e}^0_\alpha}$, which maps the previous unloaded configuration to the current state as

$$\widehat{\boldsymbol{F}}^{n+1} = \widehat{\boldsymbol{F}}^{n+1}_{\mathrm{e}^0_\alpha}\boldsymbol{F}^{n}_{\mathrm{v}_\alpha}. \tag{49a}$$

The corresponding instantaneous left Cauchy–Green tensor would be given in terms of current isochoric deformation gradient and the previous viscous right Cauchy–Green tensor as

$$\hat{\boldsymbol{b}}^{n+1}_{\mathrm{e}^0_\alpha} = \left(\widehat{\boldsymbol{F}}^{n+1}_{\mathrm{e}^0_\alpha}\right)\left(\widehat{\boldsymbol{F}}^{n+1}_{\mathrm{e}^0_\alpha}\right)^{\mathrm{T}} = \widehat{\boldsymbol{F}}_{n+1}\left(\boldsymbol{F}^{n}_{\mathrm{v}_\alpha}\right)^{-1}\left(\boldsymbol{F}^{n}_{\mathrm{v}_\alpha}\right)^{-\mathrm{T}}\widehat{\boldsymbol{F}}^{\mathrm{T}}_{n+1} = \widehat{\boldsymbol{F}}_{n+1}(\boldsymbol{C}^{n}_{\mathrm{v}_\alpha})^{-1}\widehat{\boldsymbol{F}}^{\mathrm{T}}_{n+1}. \tag{49b}$$

In the context of elastoplasticity, the above instantaneous elastic tensors are known as 'trial' state and if the resulting trial state of stresses satisfies the yield criterion, no further plastic strain will be required. In viscoelasticity, a certain amount of viscous relaxation during the increment is inevitable. This relaxation process is described in Fig. 3 by the incremental viscous deformation gradient $\boldsymbol{F}_{\Delta_\alpha}$, which maps the inelastic deformation state at stage n to its final position at step $n+1$ as

$$\boldsymbol{F}^{n+1}_{\mathrm{v}_\alpha} = \boldsymbol{F}_{\Delta_\alpha}\boldsymbol{F}^{n}_{\mathrm{v}_\alpha}. \tag{50}$$

It is now possible to link the instantaneous elastic deformation gradient, the incremental viscous deformation gradient and the current elastic deformation gradient via an incremental version of the multiplicative decomposition given as

$$\widehat{\boldsymbol{F}}^{n+1}_{\mathrm{e}^0_\alpha} = \widehat{\boldsymbol{F}}^{n+1}_{\mathrm{e}_\alpha}\boldsymbol{F}_{\Delta_\alpha}. \tag{51}$$

Note that in essence this equation is identical to expression (10). The total isochoric deformation gradient $\widehat{\boldsymbol{F}}$ has been replaced by the instantaneous elastic deformation gradient $\widehat{\boldsymbol{F}}_{\mathrm{e}^0_\alpha}$ and the incremental viscous

deformation gradient $\boldsymbol{F}_{\Delta_\alpha}$ has taken the place of the total permanent deformation $\boldsymbol{F}_{v_\alpha}$. The main advantage of Eq. (51) over Eq. (10) is that during the increment it is possible to assume that the incremental viscous deformation is co-linear (i.e., shares the same principal directions) with the elastic deformation gradient and consequently, the principal stretches of the above three tensors are related by

$$\hat{\lambda}^{e^0_\alpha}_{i\,(n+1)} = \hat{\lambda}^{e_\alpha}_{i\,(n+1)} \lambda_i^{\Delta_\alpha}. \tag{52}$$

Clearly, only in exceptional circumstances where there is no rotation of the principal directions of stress during the deformation, would the above relationship be true in terms of the total stretches $\hat{\lambda}_i$, and $\lambda_{i\alpha}$. Eq. (52) can be more conveniently expressed in terms of logarithmic stretches as

$$\ln \hat{\lambda}^{e^0_\alpha}_{i\,(n+1)} = \ln \hat{\lambda}^{e_\alpha}_{i\,(n+1)} + \ln \lambda_i^{\Delta_\alpha}. \tag{53}$$

Note that since $\hat{\boldsymbol{b}}^{n+1}_{e^0_\alpha}$ can be evaluated from Eq. (49), the instantaneous stretches in Eqs. (52) or (53) are known. In addition, as a result of the assumption that the incremental viscous deformation is co-linear with the elastic deformation, both the instantaneous elastic deformation and final elastic deformation corresponding to viscous element α are also co-linear. Consequently, the tensors $\hat{\boldsymbol{b}}^{n+1}_{e^0_\alpha}$ and $\hat{\boldsymbol{b}}^{n+1}_{e_\alpha}$ share the same set of eigenvectors $\boldsymbol{n}_{i\alpha}$ and can therefore be expressed in terms of the corresponding eigenvalues as

$$\hat{\boldsymbol{b}}^{n+1}_{e^0_\alpha} = \sum_{i=1}^{3} \left(\hat{\lambda}^{e^0_\alpha}_{i\,(n+1)} \right)^2 \boldsymbol{n}^{n+1}_{i\alpha} \otimes \boldsymbol{n}^{n+1}_{i\alpha}, \tag{54a}$$

$$\hat{\boldsymbol{b}}^{n+1}_{e_\alpha} = \sum_{i=1}^{3} (\hat{\lambda}^{e_\alpha}_{i\,(n+1)})^2 \boldsymbol{n}^{n+1}_{i\alpha} \otimes \boldsymbol{n}^{n+1}_{i\alpha}. \tag{54b}$$

3.3. Evolution equation

In order to complete the spatial viscoelastic model, it is now necessary to provide an equation for the incremental viscous strain $\lambda_i^{\Delta_\alpha}$ taking place during the interval from step n to $n+1$. In order to simplify the notation in the derivations to follow, the index $n+1$ denoting values at the current step will be dropped whenever possible. The relaxation process during the interval will now be assumed to follow a nonlinear version of Eq. (7) introduced for the linear rheological model of Section 1.1. For this purpose let s denote a time parameter varying from 0 at t_n to Δt at t_{n+1}. The rate of change of the logarithmic incremental viscous deformation is now postulated as

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\ln \lambda_i^{\Delta_\alpha}(s) \right) = \frac{1}{\tau_\alpha} \left(\ln \hat{\lambda}_i^{e^0_\alpha} - \ln \lambda_i^{\Delta_\alpha}(s) \right). \tag{55}$$

In this equation $\lambda_i^{\Delta_\alpha}(s)$ changes from 0 at $s=0$ to a final value $\lambda_i^{\Delta_\alpha}$ at $s=\Delta t$. It is therefore possible to integrate the above rate equation using a first order accurate backward Euler rule to give

$$\frac{1}{\Delta t} \ln \lambda_i^{\Delta_\alpha} = \frac{1}{\tau_\alpha} \left(\ln \hat{\lambda}_i^{e^0_\alpha} - \ln \lambda_i^{\Delta_\alpha} \right). \tag{56}$$

Re-arranging the terms in this equation enables the principal logarithmic incremental viscous strain corresponding to Maxwell element α to be evaluated in terms of the instantaneous logarithmic elastic stretch and the retardation time τ_α as

$$\ln \lambda_i^{\Delta_\alpha} = \frac{\Delta t}{\Delta t + \tau_\alpha} \ln \hat{\lambda}_i^{e^0_\alpha}. \tag{57}$$

Consequently, combining this equation with expression (53) yields the actual logarithmic elastic stretch at step $n+1$ as,

$$\ln \hat{\lambda}_i^{\mathrm{e}_\alpha} = \frac{\tau_\alpha}{\tau_\alpha + \Delta t} \ln \hat{\lambda}_i^{\mathrm{e}_\alpha^0}. \tag{58}$$

Interestingly, for the particular case of the simple material defined by Eqs. (47a) and (47b)–(48a) and (48b), an identical relationship to that obtained for the second Piola–Kirchhoff tensor in Section 2.5 is now obtained for the Cauchy stress components as

$$\sigma'_{i\alpha} = \frac{\tau_\alpha}{\tau_\alpha + \Delta t} \tilde{\sigma}'_{i\alpha}, \quad \tilde{\sigma}'_{i\alpha} = \sigma'_{i\alpha}(\hat{\lambda}_i^{\mathrm{e}_\alpha^0}), \tag{59}$$

where the notation $\tilde{\boldsymbol{\sigma}}'$ indicates instantaneous stresses obtained under the preliminary assumption that there is no further viscous strain during the increment. Additionally, it is worth mentioning that a more accurate time integration of Eq. (55) leads to the relationship

$$\ln \hat{\lambda}_i^{\mathrm{e}_\alpha} = \mathrm{e}^{-\Delta t/\tau_\alpha} \ln \hat{\lambda}_i^{\mathrm{e}_\alpha^0}. \tag{60}$$

The use of this expression in place of Eq. (58) will have to be explored in the context of a computational implementation of the above model.

Finally, once the current elastic stretches are evaluated with the help of Eqs. (58) or (60), the elastic left Cauchy–Green tensor $\hat{\boldsymbol{b}}_{\mathrm{e}_\alpha}$ can be obtained with the help of Eq. (54b) and the internal variables $\boldsymbol{C}_{\mathrm{v}_\alpha}$ can be updated as

$$\left(\boldsymbol{C}_{\mathrm{v}_\alpha}^{n+1}\right)^{-1} = \left(\boldsymbol{F}_{\mathrm{v}_\alpha}^{n+1}\right)^{-1} (\boldsymbol{F}_{\mathrm{v}_\alpha})^{-\mathrm{T}} = \widehat{\boldsymbol{F}}_{n+1}^{-1} \hat{\boldsymbol{b}}_{\mathrm{e}_\alpha}^{n+1} \widehat{\boldsymbol{F}}_{n+1}^{-\mathrm{T}}. \tag{61}$$

3.4. Tangent modulus

In order to complete the spatial version of the proposed viscoelastic model, the tangent material operator needs to be derived. A simple procedure required for this derivation when the strain energy is defined in terms of principal stretches is explained in details by Ogden (1984) and Bonet and Wood (1997) (Refs. Simo and Taylor, 1991 or Miehe, 1994 for an alternative procedure). In summary, the tangent operator is split into volumetric, long term and viscous components as

$$\mathbf{c} = \mathbf{c}_{\mathrm{vol}} + \hat{\mathbf{c}}_\infty + \sum_\alpha \hat{\mathbf{c}}_\alpha. \tag{62}$$

The volumetric component is independent of the viscoelastic model and can be obtained by pushing forward Eq. (34) for the Lagrangian version to give

$$\mathbf{c}_{\mathrm{vol}} = (KJ + p)(\boldsymbol{I} \otimes \boldsymbol{I}) - 2p\mathbf{i}, \tag{63a}$$

$$K = U''(J), \tag{63b}$$

where $\mathbf{i}$ denotes the fourth order identity tensor defined by

$$2\mathrm{i}_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}. \tag{64}$$

The long term component is evaluated following the procedure given by Bonet and Wood (1997) to yield

$$\hat{\mathbf{c}}_\infty = \sum_{i,j=1}^{3} \frac{1}{J} \frac{\partial (J\sigma'_{i\infty})}{\partial \ln \lambda_j} \boldsymbol{n}_i \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_j - \sum_{i=1}^{3} \sigma'_{i\infty} \boldsymbol{n}_i \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_i + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \frac{\sigma'_{i\infty}\lambda_j^2 - \sigma'_{j\infty}\lambda_i^2}{\lambda_i^2 - \lambda_j^2} (\boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_i). \tag{65}$$

Similarly, the viscous components are given by the fourth order tensor defined in terms of principal direction and derivatives with respect to the instantaneous elastic stretches as

$$\hat{\mathbf{c}}_\alpha = \sum_{i,j=1}^{3} \frac{1}{J} \frac{\partial (J\sigma'_{i\alpha})}{\partial \ln \lambda_j^{e_\alpha^0}} \boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{j\alpha} \otimes \boldsymbol{n}_{j\alpha} - \sum_{i=1}^{3} \sigma'_{i\alpha} \boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{i\alpha} + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \frac{\sigma'_{i\alpha}(\hat{\lambda}_j^{e_\alpha^0})^2 - \sigma'_{j\alpha}(\hat{\lambda}_i^{e_\alpha^0})^2}{(\hat{\lambda}_i^{e_\alpha^0})^2 - (\hat{\lambda}_j^{e_\alpha^0})^2} (\boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{j\alpha} \otimes \boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{j\alpha} + \boldsymbol{n}_{i\alpha} \otimes \boldsymbol{n}_{j\alpha} \otimes \boldsymbol{n}_{j\alpha} \otimes \boldsymbol{n}_{i\alpha}). \tag{66}$$

As an illustration consider the simple material defined by Eqs. (47a) and (47b), (48a) and (48b). In this case the above derivatives are easily obtained as (Bonet and Wood, 1997),

$$\frac{\partial (J\sigma'_{i\infty})}{\partial \ln \lambda_j} = 2\mu \left(\delta_{ij} - \frac{1}{3} \right), \tag{67a}$$

$$\frac{\partial (J\sigma'_{i\alpha})}{\partial \ln \lambda_j^{e_\alpha^0}} = \frac{2\mu\beta_\alpha\tau_\alpha}{\tau_\alpha + \Delta t} \left(\delta_{ij} - \frac{1}{3} \right). \tag{67b}$$

4. Concluding remarks

The paper has presented new Lagrangian and spatial formulations for large strain isothermal viscoelasticity as extensions of the well known generalized linear Maxwell model. Both formulations proposed are based on the multiplicative decomposition of the deformation gradient now common and fully accepted in large strain elastoplasticity. The Lagrangian formulation is similar to the model proposed by Lubliner (1985) except for the use of a nonlinear relaxation equation for the internal variables. This equation is derived from a linear relaxation model which gives the rate of change of viscous stress components *at constant total strain* in terms of a retardation time parameter and the current state of stress. It is simply an extension of a similar equation governing the relaxation of the forces in the Maxwell dashpot at constant total strain. The resulting rate equations for the plastic strain can be seen as a particular case of the general elastoplastic formulation proposed by Simo (1988). Crucially, the resulting relaxation equation can be interpreted incrementally in a way that leads to a trivial evaluation of the second Piola–Kirchhoff stresses in terms of the stresses that would be measured if the increment had taken place instantaneously. Given the stresses, the model will then require the evaluation of the updated viscous strains. In general, this process will not be trivial as it requires the solution of a set of six coupled nonlinear equations. However, a local Newton–Raphson procedure will furnish the solution at a reasonable amount of effort. In order to illustrate the formulation, the simple neo-Hookean model has been used to derive the resulting equations for the second Piola–Kirchhoff stress and the tangent operator.

The spatial model proposed for isotropic materials is equally based on the same multiplicative decomposition. However, a different evolution equation is postulated in terms of the logarithmic stretches of the elastic and incremental viscous deformation. The incremental, rather than total, viscous deformation is used for this purpose because only the former that can be assumed to be co-linear with the elastic strain for

isotropic materials. The simple linear Maxwell relaxation equation is now extended to the nonlinear case by re-writing it in terms of logarithmic stretches. The resulting formulation only coincides with the previous Lagrangian model for a particular case of elastic strain energy function.

For simplicity only first order accurate integration procedures have been discussed in this paper. However both the Lagrangian and spatial formulations can be integrated more accurately. This aspect of the formulation will have to be explored in the context of a finite element formulation where the gains of more accurate integration can be measured against other problem such as ease of implementation or convergence of the nonlinear equations.

References

Bonet, J., Wood, R.D., 1997. Nonlinear Continuum Mechanics for Finite Element Analysis. Cambridge University Press, Cambridge.

Bonet, J.A., Burton, A., 1998. A simple orthotropic, transversely isotropic hyperelastic constitutive equation for large strain computations. Comput. Meth. Appl. Mech. Engng. 162, 151–164.

Findley, W.N., Lai, J.S., Onaran, K., 1976. Creep and Relaxation of Nonlinear Viscous Materials. Dover Publications, UK.

Govindjee, S., Simo, J.C., 1992. Mullins effect and the strain amplitude dependence of the elastic modulus. Int. J. Solids Structures 29, 1737–1751.

Green, M.S., Tobolsky, A.V., 1946. A new approach to the theory of relaxing polymeric media. J. Phys. Chem. 14, 80–92.

Holzapfel, G.A., Reiter, G., 1995. Fully coupled thermomechanical behaviour of viscoelastic solids treated with finite elements, Int. J. Engng. Sci. 33, 1037–1058.

Holzapfel, G.A., 1996. On large strain viscoelasticity: continuum formulation and finite element applications to elastometric structures. Int. J. Num. Meth. Engng. 39, 3903–3926.

Holzapfel, G.A., Eberlein, R., Wriggers, P., Weizsacker, H., 1996. Large strain analysis of soft biological membranes: formulation and finite element analysis. Comput. Meth. Appl. Mech. Engng. 132, 45–61.

Holzapfel, G.A., Simo, J.C., 1996. A new viscoelastic constitutive model for continuous media at finite thermomechanical changes. Int. J. Solids Structures 33, 3019–3034.

Lion, A., 1997. A physically based method to represent the thermo-mechanical behaviour of elastometers. Acta Mechanica 123, 1–25.

Lubliner, J., 1985. A model of rubber viscoelasticity. Mech. Research Comm. 12, 93–99.

Miehe, C., 1994. Aspects of the formulation and finite element implementation of large strain isotropic elasticity. Int. J. Num. Meth. Engng. 37, 1981–2004.

Ogden, R.W., 1984. Non-linear Elastic Deformations. Ellis Horwood, Chichester.

Peric, D., Owen, D.R.J., Honnor, M.E., 1992. A model for large strain elastoplasticity based on logarithmic strains: computational issues. Comput. Meth. Appl. Mech. Engng. 94, 35–61.

Reese, S., Govindjee, S., 1998. A theory of finite viscoelasticity and numerical aspects. Int. J. Solids Structures 35, 3455–3482.

Simo, J.C., Taylor, R.L., Pister, K.S., 1985. Variational and projection methods for the volume constraint in finite deformation elastoplasticity. Comput. Meth. Appl. Mech. Engng. 51, 177–208.

Simo, J.C., 1987. On a fully three dimensional finite strain viscoelastic damage model: formulation and computational aspects. Comp. Meth. Appl. Mech. Engng. 60, 153–173.

Simo, J.C., 1988. A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: part 1. Continuum formulation. Comput. Meth. Appl. Mech. Engng. 66, 199–219.

Simo, J.C., Taylor, R.L., 1991. Quasi incompressible finite elasticity in principal stretches. Continuum basis and numerical algorithms. Comput. Meth. Appl. Mech. Engng. 85, 273–310.

Simo, J.C., 1992. Algorithms for static and dynamic multiplicative plasticity that preserve the classical return mapping schemes of the infinitesimal theory. Comput. Meth. Appl. Mech. Engng. 99, 61–112.

Simo, J.C., Hughes, T.J.R., 1998. Computational Inelasticity. Springer, New York.